Note on the derived logarithmic fuction of a multiplicative arithmetic function

Michinori YAMASHITA[†] Daisuke MIYATA[‡] Kouichi SHIBAKI[*]

[†]Rissho University, [‡]Chiba University of Commerce,
[*]Iwate Prefectural University, Miyako Junior College

2023/09/12@IAR2023

Agenda of talk

We have studied various behaviours of Euler functions, among which we have defined the derived logarithmic function L of the Euler function φ, investigated various properties of L and L, φ, and proposed some problems such as the abc-conjecture (unresolved) on L.[9, 20, 23].

The abc conjecture on L

The following inequality holds for $abc-$triple.

$$\max(L(a),\ L(b),\ L(c)) \leqq 2L(\mathrm{rad}\ (abc))$$

Verified for $n \leqq 5 \cdot 10^8$. More than two present except for $n = 9,\ 25,\ 121$, a stronger conjecture than Brocard's.

There exists a prime between $\varphi(n)$ and n.

For natural numbers $n > 1$, there are primes in the interval $(\varphi(n),\ n]$.

By the way, Euler's iteration of the φ function is well known for the iteration of number-theoretic functions.
It is noted that if "$n > 1$" for the φ function, then $\varphi(n) < n$", and if we write $\varphi^k(n) = \varphi(\varphi^{k-1}(n)) \quad (k > 1)$.

$$\varphi(n),\ \varphi(\varphi(n)),\ \varphi(\varphi(n)),\ \cdots,\ \varphi^{\ell}(n),\ \cdots$$

and φ acting finitely many times shows that for any natural number $n > 1$, there always exists some natural number m such that $\varphi^m(n) = 1$.

The nature of m was first investigated by S.S.Pillai (1929) and then by H.Shapiro (1943), and without knowing the results of H.Shapiro (partly due to the background of insufficient literature exchange), it was not until 1954-1955 that the results of H.Lindgren and E.S.Barnes (1955-1955) and Murr?anyi (1960) were studied in more detail. H.Lindgren, E.S.Barnes and Mur?anyi?citeL1,B in 1954-1955 and again in 1960, without knowledge of H.Shapiro's results.

Also, inspired by the results of H. Shapiro, P. Erdös, A. Granvilie, C. Pomerance, C. Spiro, P. Pollack and many other mathematicians have investigated the average of $\varphi^k(n)$, $\sum \varphi^k(n)$, $\prod \varphi^k(n)$, $\frac{\varphi^k}{\varphi^{k+1}}(n)$ and other properties are investigated

The content of the iteration which S.S.Pillai started to investigate can be summarised by the following result. For a natural number x, the smallest integer m such that $\varphi^m(x) = 2$ is denoted by $C(x)$ according to H.Shapiro[14] and $C(1) = C(2) = 0$. In this case, we have

Theorem

(Shapiro[14, 16]) *For any natural number $x,\ y$.*

$$C(xy) = C(x) + C(y) + \epsilon(x, y)$$

holds, where

$$\epsilon(x, y) = \begin{cases} 1 & (x,\ y \text{ both even}) \\ 0 & (\text{ otherwise }) \end{cases}$$

Later, in 1950, H.Shapiro[15] further defined the following number theoretic function for the prime factorisation of $n = \prod_{i=1}^{r} p_i^{e_i}$ for $n > 1$, which is

Definition

*(*Shapiro[15]*)*

$$f(n) = \prod_{i=1}^{r} f(p_i)(A(p_i))^{e_i - 1}$$

However, $f(p_i),\ A(p_i)$ *is a natural number such that* $0 < f(p_i) < p_i,\ 0 < A(p_i) \leqq p_i,\ f(2) = 1,\ A(2) = 2.$

H.Shapiro[15] gives $f(p_i)$ with
$f(p_i) = p_{i-1} \quad (i > 1),\ f(p_1) = f(2) = 1$ as a simple example[16].

Using this f, $k_f(1) = k_f(2) = 0$ for $k = k_f(x)$ such that $f^k(x) = 2$ for $x > 2$, the following results were obtained.

Theorem

(Shapiro[15, 16, **?**]) *If* $c_f(x)$ *is defined as*

$$c_f(x) = \begin{cases} k_f(x) & x : \text{odd} \\ k_f(x) + 1 & x : \text{even} \end{cases}$$

A logarithmic relational expression holds for $c_f(x)$*.*

$$c_f(xy) = c_f(x) + c_f(y)$$

As we have seen from the above, if we are not fixated on the number of times to reach 1 or 2 in the quasi-logarithmic relational expression for $C(x)$ above, or on the evenness of n as in $c_f(x)$, we can review $C(x),\ c_f(x)$ and define $L(x)$ as follows, which is fully logarithmic relation is obtained.
This was also noticed by H. Lindgren[6] and Yamashita[19].
The essential part remains the same, but there are subtle differences in how logarithmic/quasi-logarithmic relations are handled and generalised, described as follows.

Definition

Define $L(x)$ as follows

$$L(x) = \begin{cases} L(1) = 0 & \\ L(\varphi(x)) & (\text{ } x \text{ is odd}) \\ L(\varphi(x)) + 1 & (\text{ } x \text{ is even}) \end{cases}$$

In this case, for any natural number $x,\ y$, we have

$$L(xy) = L(x) + L(y)$$

holds.

On the other hand, Miyata-Yamashita [8] discards even-oddness and obtains the following theorem by defining f as follows.

Before that, let us extend the Euler function φ from a different point of view.

Definition

(Miyata - Yamashita[8]) *Let $\mathcal{P}$, $\mathcal{N}$ be the set of prime numbers and the set of natural numbers, respectively, and the function $f : \mathcal{P} \longrightarrow \mathcal{N}$ satisfy $1 \leqq f(p) < p \quad (\, p \in \mathcal{P}\,)$. In this case, the Eulerian function $\varphi_f(x)$ that depends on f is*

$$\varphi_f(x) = x\prod_{i=1}^{r}\frac{f(p_i)}{p_i} \qquad \left(\varphi(x) = x\prod_{i=1}^{r}\frac{(p_i-1)}{p_i}\right)$$

where $x = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$.

Using the φ_f dependent on f, the function L_f is defined as

Theorem

(Miyata-Yamashita[8])

$$L_f(1) = 0$$
$$L_f(x) = L_f(\varphi_f(x)) + \# \left| \{ p \in f^{-1}(1) \mid p|x \} \right|$$

Then, for any natural number x, y, we have

$$L_f(xy) = L_f(x) + L_f(y)$$

holds.

is obtained. This L_f gives rise to a logarithmic function of even-oddness discarded form.

Also, for $L(x)$ up to now, the following simple evaluation has been obtained.

Proposition

(Shapiro, et al.[14, 10, 8, 20])

$$\log_3 x \leqq L(x) \leqq \log_2 x$$

$\psi_0$function and its derived logarithmic function L_{ψ_0}

Let us now define the ψ_0 function, a variant of Dedekind's ψ function. As $n = \prod p_i^{e_i}$.

Definition

$$\psi_0(n) = \begin{cases} 1 & (\ n = 1\ ) \\ \prod_i p_i^{e_i - 1} \left\lfloor \frac{p_i + 1}{2} \right\rfloor & (\ n > 1\ ) \end{cases}$$

The ψ_0 function values for primes less than or equal to 113 are as follows.

Table of ψ_0(prime) from 2 to 113

p	2	3	5	7	11	13	17	19	23	29
ψ_0	1	2	3	4	6	7	9	10	12	15
p	31	37	41	43	47	53	59	61	67	71
ψ_0	16	19	21	22	24	27	30	31	34	36
p	73	79	83	89	97	101	103	107	109	113
ψ_0	37	40	42	45	49	51	52	54	55	57

Our function ψ_0 satisfies the assumption on f in our theorem. The value of L_{ψ_0} derived from ψ_0 under 100 is as follows.

Table of L_{ψ_0} from 1 to 100

n	1	2	3	4	5	6	7	8	9	10
L_{ψ_0}	0	1	1	2	1	2	2	3	2	2
n	11	12	13	14	15	16	17	18	19	20
L_{ψ_0}	2	3	2	3	2	4	2	3	2	3
n	21	22	23	24	25	26	27	28	29	30
L_{ψ_0}	3	3	3	4	2	3	3	4	2	3
n	31	32	33	34	35	36	37	38	39	40
L_{ψ_0}	4	5	3	3	3	4	2	3	3	4
n	41	42	43	44	45	46	47	48	49	50
L_{ψ_0}	3	4	3	4	3	4	4	5	4	3
n	51	52	53	54	55	56	57	58	59	60
L_{ψ_0}	5	4	3	4	3	5	3	3	3	4
n	61	62	63	64	65	66	67	68	69	70
L_{ψ_0}	4	5	4	6	3	4	3	4	4	4
n	71	72	73	74	75	76	77	78	79	80
L_{ψ_0}	4	5	2	3	3	4	4	4	4	5
n	81	82	83	84	85	86	87	88	89	90
L_{ψ_0}	4	4	4	5	3	4	3	5	3	4
n	91	92	93	94	95	96	97	98	99	100
L_{ψ_0}	3	5	4	5	3	6	4	5	4	4

For L_{ψ_0}, as for $L(x)$, our abc-conjecture (verified for $c < 2^{30}$) is considered ($c < 2^{30}$, [9]), but for L_{ψ_0}, each value of abc–triple etc. in the range $c < 10^5$ on a PC experiments show that four counterexamples already exist, as follows. Therefore, the conjecture does not hold as it is, but further experiments are needed to see if it is possible to replace "2" on the right-hand side of the abc-conjecture for L by a larger number. (In the PC experiment, it holds for "3".)

a	b	c	$L_{\psi_0}(\text{rad } abc)$	$L_{\psi_0}(a)$	$L_{\psi_0}(b)$	$L_{\psi_0}(c)$
3	5^3	2^7	3	1	3	7
3^5	$5 * 19^2$	2^{11}	4	5	6	11
7	181^2	2^{15}	7	2	8	15
7^3	3^{10}	$2^{11} * 29$	6	6	10	14

Programs checked (Python)

```
from math import gcd

N = 10**5

f = [i for i in range(N)]
rad = [1]*N
Lf = [0] * N

for i in range(2, N):
    if f[i] == i:
        for j in range(i, N, i):
            f[j] = f[j]//i * ((i+1)//2)
            rad[j] *= i

for i in range(2, N):
    if i % 2 == 0:
        Lf[i] = Lf[f[i]] + 1
    else:
        Lf[i] = Lf[f[i]]
```

```
for c in range(1, N):
    for a in range(1, c//2+1):
      if gcd(a,c) == 1:
        b = c - a
        q = Lf[c]/(Lf[rad[a]]+
                   Lf[rad[b]]+
                   Lf[rad[c]])
        if q > 2:
             print((a,b,c), q)
```

Also, for the evaluation of L_{ψ_0} corresponding to the proposition3.7 from above

$$L_{\psi_0}(x) \leqq \log_2 x$$

is obvious, but if we consider the evaluation from below by mimicking $L(x)$, $L_{\psi_0}(x) = 1$, since x such that $x = 2,\ 3,\ 5$, we obtain

$$L_{\psi_0}(x) \leqq n \Leftrightarrow x \leqq 5^n \Leftrightarrow \log_5 x \leqq L_{\psi_0}(x)$$

is expected, e.g.

$$L_{\psi_0}(73) = L_{\psi_0}(37) = 2 \ \text{ when } \quad 37,\ 73 \ \nleqq 5^2$$

No evaluation from below for $L_{\psi_0}(x)$ is currently available, as the easy conjecture does not hold,

Issues for the future

- How far up the "2" on the right-hand side can be raised by the abc conjecture for L_{ψ_0}?
- Computation of the Dirichlet series $L(s, \psi_0)$ of ψ_0.

$$L(s,\ \psi) = \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)}$$

  Can it be expressed using $\zeta(s)$ in the same way as
- Determination of the valuation from below for $L_{\psi_0}(x)$.

Thanks

Thank you for your kind attention.

Barnes E. S., *Note 225*, Australian Math. Teacher 11 (1955), 20-21

Erdös, P., Granvilie A., Carl Pomerance, C., Spiro, C.,*On the Normal Behavior of the Iterates Of some Arithmetic Functions*, Analytic Number Theory (Allerton Park, IL, 1989) (eds. Berndt, B. C., Diamond, H. G., Halberstam, H. and Hildebrand, A.), Progress in Mathematics, 85 (Birkhäuser, Boston, MA, 1990), 165?204

Hirayama, A., *Yoshihiro KURUSHIMA* (in Japanese), J. of JSHM, Japanese Society for the History of Mathematics, No.108, (1986) 1-27 `http://www.wasan.jp/sugakusipdf/sugakusi108.pdf` (Ref. 2023.5.18)

Kato, H., *History of Pre-Meiji Japanese Mathematics: Achievements of Yoshihiro Kurushima* (in Japanese), Japan Academy, ed. (1957) 62–76

Kato, H., *Number Theory (Study of Japanese mathematics)*, Japan Society for the Promotion of Science, 1964.3.15, 117

- Lindgren, H., *Note.211 Given the Method, Find the Problem*, Australian Math. Teacher 10 (1954), 52-53
- Lind D., *Problem 5239*, Amer. Math. Monthly, 71 (1964), p.1047, Solution., Ibid., 72 1965), p.1035.
- Miyata, D., Yamashita, M., *Note on derived logarithmic functions of Euler's functions* (in Japanese), Proceedings of Autum meeting(App. Math.), Math. Soc. of Japan, 2004.9, J. of IIARS, Vol.1-1（2017.6）, 111-116
- Miyata, D. , Yamashita, M., *Fast enumeration algorithm for* $abc-$*triple of derived logarithmic functions of Euler functions*,
- Murányi, A., *Az Euler-félé* ϕ *-függvény iterálásával nyert számelméleti függvényröl*, Mat. Lapok (Budapest) 11 (1960), 46-67
- Pillai S. S., *On some functions connected with* $\phi(n)$, Bull. Amer. Soc., 35 (1929), 832-836
- Pillai S. S., *On a function connected with* $\phi(n)$, Bull. Amer. Soc., 35 (1929), 837-841

- Pollack, P., *Two remarks on iterates of Euler's totient function*, Arch. Math. (Basel) 97(5) (2011), 443?452
- Shapiro, H., *An arithmetic function arising from the ϕ function*, Amer. Math. Monthly, 50 (1943), 18-30
- Shapiro, H., *On the iterates of a certain class of arithmeti functions*, Comm. Pure Appl. Math. 3 (1950), 259-272
- Shapiro, H., *Introduction to the Theory of Numbers*, John Wiley & Sons, New York et al., 1983,:3.Arithmetic Functions § 3.7 Exercise # 17, 77-78
- Shparlinski, I. E., *On the sum of iterations of the Euler function*, J. Integer Seq. 9(1) (2006), Article no. 06.1.6.
- Tooyama, H., *Elementary Number Theory* (in Japanese), NIPPON HYORON SHA Co.,Ltd., 1972.1.1.
- Uchiyama Saburo (Okayama univ.) → Yamashita Michinori, private communication, 1977.9.12

http://yamashita-lab.net/yamasita-diary/uchiyama_19770912.pdf

- Yamashita, M.–Miyata, D., *On the abc conjecture for a derived logarithmic function of the Euler function*, Proceedings of1st CCATS2015_IEEE(International Conference on Computer Application & TechnologieS 2015), Session # 7(9.2), Kunibiki Messe(Matsue，2015.8.31）9–2
- Yamashita, M.–Miyata, D. –Fujita, N., *The abc conjecture using logarithmic functions derived of Euler's function and its computer verification*，The Rissho Int'l J.，vol.2-1（2019.3），275-291
- Yamashita, M., Miyata, D., Fujita, N. , *Aspects of the derived logarithmic function* $L(x)$ *of the Euler function* (in Japanese), Bulletin of the Faculty of Geo-Environmerntal Science, Rissho Univ., Vol.20（2021.3），67-72
- Yamashita, M.–Miyata, D., *20220321 notes:On a certain conjecture for* $\varphi(n)$, (unpublished), 2022.0321.
http://yamashita-lab.net/yamasita-diary/20220321

`memo.pdf`

Yamashita, M.–Miyata, D.–Shgibaki, K., *Note on the derived logarithmic fuction of a multiplicative arithmetic function*, 1st International Conference on ICT Application Research (IAR2023), Proc. of IAR2023 (Session 4: Mathematics and ICT Application Research), Fukui, 2023.09.10-09.12 (to appear)