Environmental Info-Mathematics I [1H105500]

Final Examination

Examiner: YAMASHITA M.

Date: Tuesday, January 18, 2011

Time: 10:35 A.M. – 11:40 A.M.

Read the directions to each problem carefully. Show your work and answer in complete sentences when appropriate. This exam has 18 questions on 5 pages including this cover.

Problem 0. (5 points each) Describe the following terms.

(1) propostion
(2) theorem
(3) lemma
(4) proof
(5) q.e.d.
(6) c.q.f.d.
(7) false
(8) negation
(9) Law of the excluded middle
(10) Russel's paradox
(11) truth table
(12) logical product
(13) logical sum
(14) assumption
(15) equivalent
(16) tautology
(17) inverse of statement
(18) converse of statement
(19) contraposition of statement
(20) propositional function
(21) set
(22) subset
(23) family of sets
(24) countable sets / enumerable sets
(25) cardinality of a infinite set
(26) empty set $\emptyset$]
(27) one-to-one mapping / injection
(28) onto mapping / surjection
(29) supremum
(30) lower bound
(31) maximum
(32) continuous
(33) intermediate value thorem

(34) extrem value

(35) inflexion point

Problem 1. (5 points each) Determine whether each statement is true or false. If the statement is false, explain why.

(1) $\emptyset \in \{$ Mexico, United States, Canada $\}$

(2) $\emptyset \subseteq \{$ Miki, Akiha, Noriko, Tamami, Kotoko , Momoko $\}$

(3) The set $\{ 1, 2, 3, \ldots , 100 \}$ has 2^{100} proper subsets.

(4) $\emptyset \nsubseteq \{ \emptyset, \{ \emptyset \}\}$

(5) $\emptyset \subseteq \{ \emptyset \}$

(6) $\emptyset \in \{ \emptyset \}$

(7) $A = -\int_C x\ dy$, where a is the area of the region bounded by C and C is counterclockwise.

(8) $A = -\int_C y\ dx$, where a is the area of the region bounded by C and C is counterclockwise.

(9) The statement that if A then B is equivalent to the statement that if ¬B then ¬A.

(10) The Maclaurin series associate with e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{, where } 0!=1.$$

(11) The Maclaurin series associate with $\cos x$ is

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

Problem 2. (30 points) In a certain town, a barber shaves all those men and only those men who do not shave themselves. Consider each of the following sets:

$A = \{ x |\ x$ is a man of the town who shaves himself$\}$

$B = \{ x |\ x$ is a man of the town who does not shave himself$\}$

The one and only barber in the town is Sweeney Todd. If s represents Sweeny Todd.

(1) is $s \in A$?

(2) is $s \in B$?

Problem 3. (20 points) Let P, Q be propositions (or statements). Show that $P \Longrightarrow Q$ and $\neg P \vee Q$ are equivalent, using a truth table.

Problem 4. (7 points each) Evaluate the following limits or show that it does not exist.

(1) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

(2) $\lim_{x \to 0} \frac{\sin 3x}{x + x^3}$

(3) $\lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$

(4) $\lim_{x \to 0+} x^x$

(5) $\lim_{x \to 0} \sin \frac{1}{x}$

(6) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

(7) $\lim_{x \to 0} \frac{\sin^2 x}{x}$

(8) $\lim_{x \to 0} \frac{\sin x^2}{x^2}$

(9) $\lim_{x \to 2} \frac{\sqrt{x^2 + 3x + 1}}{\sqrt{16x^2 + x + 2}}$

Problem 5. (15 points) Find the derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the following functions.

(1) $f(x, y) = \arctan\left(\frac{y}{x}\right)$

(2) $f(x, y) = y^3 + 3xy + x^3$

Problem 6. (5 points each) Evaluate the following integrals.

(1) $\int \frac{1 - \sin x}{x + \cos x} dx$

(2) $\int \frac{dx}{x^2 + 4x + 3}$

(3) $\int x \sin x \, dx$

(4) $\int e^x \, dx$

(5) $\int \log 3x \, dx$

Problem 7. (50 points) Show that the cardinality of $\mathbf{R}$ is uncountable, where $\mathbf{R}$ is the set of real numbers.

Problem 8. (10 points) Express the given numbers in the form $re^{i\theta}$ for a positive real number r and argument θ, $-\pi < \theta < \pi$.

$$5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \cdot 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Problem 9. (10 points each) Evaluate the followings with minimum effort, C is a counterclockwise curve bounding a region of area 5.

(1) $\oint_C 3ydx$

(2) $\oint_C (2ydx + 6xdy)$

Problem 10. (20 points) Evaluate the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ using the formula $A = \displaystyle\int_C x\ dy$, where a is the area of the region bounded by C and C is counterclockwise.

Problem 11. (15 points each) Evaluate the following definite integrals or show divergence.

(1) $\displaystyle\int_0^1 x \log x\ dx$

(2) $\displaystyle\int_0^\infty x^2 e^{-5x}\ dx$

Problem 12. (40 points) Let f be

$$f(x) = \begin{cases} x & x \text{ is rational,} \\ 0 & \text{otherwise.} \end{cases}$$

Is $f(x)$ continuous at 1? Is $f(x)$ continuous at 0? Explain.

Problem 13. (10 points) Compute $\displaystyle\int_C xy\ dx$, where C is the straight line from (0, 0) to (2, 2).

Problem 14. (25 points) Let a_n be a sequence for which $a_0 = 0,\ a_1 = 1$. Find a_n if $\{ a_n \}$ is given by recursion formula

$$a_{n+2} - a_{n+1} - 6a_n = 0$$

Problem 15. (25 points) Solve the following differntial equation.

$$y'' - y' - 6y = 0$$

Problem 16. (40 points) Find the Fourier coefficients of the square wave function

$$w(x) = \begin{cases} 1 & 0 \le x \le \pi, \\ -1 & -\pi < x < 0. \end{cases}$$

Problem 17. (30 points) Find the Fourier coefficients of the following function

$$f(x) = \begin{cases} x & 0 \le x \le \pi, \\ -x & -\pi < x < 0. \end{cases}$$