Fundamental (Row / Column) Operation

The following operations are said to be **fundamental row (resp. column) operation** for matrices.

(1) interchanging the ith and jth row (resp. column). $\mathrm{R}_i \leftrightarrow \mathrm{R}_j$ (resp. $\mathrm{C}_i \leftrightarrow \mathrm{C}_j$)

(2) multipling the ith row (resp. column) by a scalar α. $\mathrm{R}_i \leftarrow \alpha\mathrm{R}_i$ (resp. $\mathrm{C}_i \leftarrow \alpha\mathrm{C}_i$)

(3) adding the ith row (resp. column) to the jth row (resp. column). $\mathrm{R}_i \leftarrow \mathrm{R}_i+\mathrm{R}_j$ (resp. $\mathrm{C}_i \leftarrow \mathrm{C}_i+\mathrm{C}_j$)

Sometimes "fundamental operation" is called "elementary operation".

1 Examles

1.1 Example 1

$$\begin{matrix} R_1 & \rightarrow \\ R_2 & \rightarrow \\ R_3 & \rightarrow \end{matrix} \begin{pmatrix} 0 & 2 & 4 & -2 \\ 3 & -9 & 6 & 3 \\ -2 & 6 & -4 & -2 \end{pmatrix}$$

$$\longrightarrow \left\{\mathrm{R}_1 \leftrightarrow \mathrm{R}_2\right\} \longrightarrow \begin{pmatrix} 3 & -9 & 6 & 3 \\ 0 & 2 & 4 & -2 \\ -2 & 6 & -4 & -2 \end{pmatrix} \longrightarrow \left\{\mathrm{R}_1 \leftarrow \frac{1}{3}\mathrm{R}_1\right\} \longrightarrow \begin{pmatrix} 1 & -3 & 2 & 1 \\ 0 & 2 & 4 & -2 \\ -2 & 6 & -4 & -2 \end{pmatrix}$$

$$\longrightarrow \left\{ \mathrm{R}_2 \leftarrow \frac{1}{2}\mathrm{R}_2\right\} \longrightarrow \begin{pmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ -2 & 6 & -4 & -2 \end{pmatrix} \longrightarrow \left\{ \mathrm{R}_3 \leftarrow \mathrm{R}_3 + 2\mathrm{R}_1\right\} \longrightarrow \begin{pmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \left\{\mathrm{R}_1 \leftarrow \mathrm{R}_1 + 3\mathrm{R}_2\right\} \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 8 & -2 \\ 0 & 1 & 2 & -1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

1.2 Example 2

$$\begin{matrix} & \\ & \\ R_1 & \rightarrow \\ R_2 & \rightarrow \\ R_3 & \rightarrow \end{matrix} \begin{matrix} \mathrm{C}_1 & \mathrm{C}_2 & \mathrm{C}_3 & \mathrm{C}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 2 & 4 & -2 \\ 3 & -9 & 6 & 3 \\ -2 & 6 & -4 & -2 \end{matrix}$$

$$\longrightarrow \left\{ \mathrm{R}_1 \leftarrow \frac{1}{2}\mathrm{R}_1 \right\} \longrightarrow \begin{pmatrix} 0 & 1 & 2 & -1 \\ 3 & -9 & 6 & 3 \\ -2 & 6 & -4 & -2 \end{pmatrix} \longrightarrow \left\{ \mathrm{R}_2 \leftarrow \frac{1}{3}\mathrm{R}_2 \right\} \longrightarrow \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & -3 & 2 & 1 \\ -2 & 6 & -4 & -2 \end{pmatrix}$$

$$\longrightarrow \left\{ \mathrm{R}_3 \leftarrow \frac{-1}{2}\mathrm{R}_3 \right\} \longrightarrow \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & -3 & 2 & 1 \\ 1 & -3 & 2 & 1 \end{pmatrix} \longrightarrow \left\{ \mathrm{C}_1 \leftrightarrow \mathrm{C}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & -1 \\ -3 & 1 & 2 & 1 \\ -3 & 1 & 2 & 1 \end{pmatrix}$$

$$\longrightarrow \left\{ \mathrm{R}_3 \leftarrow \mathrm{R}_3 - \mathrm{R}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & -1 \\ -3 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \left\{ \mathrm{R}_2 \leftarrow \mathrm{R}_2 + 3\mathrm{R}_1 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 8 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \left\{ \text{C}_3 \leftarrow \text{C}_3 - 2\text{C}_1 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 8 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \left\{ \text{C}_4 \leftarrow \text{C}_4{+}\text{C}_1 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \left\{ \text{C}_3 \leftarrow \text{C}_3 - 8\text{C}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \left\{ \text{C}_4 \leftarrow \text{C}_4 + 2\text{C}_2 \right\} \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right)$$

1.3 Example 3 (inverse matrix 2×2)

1.3.1 Row operation

Let $\begin{matrix} R_1 & \rightarrow \\ R_2 & \rightarrow \end{matrix} \left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array} \right) = \left(\begin{array}{c|c} A & E_2 \end{array} \right)$ be an augmented matrix of $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ for finding A^{-1}.

$$\longrightarrow \left\{ \text{R}_1 \leftarrow 2\text{R}_1 \right\} \longrightarrow \begin{pmatrix} 6 & 2 & 2 & 0 \\ 5 & 2 & 0 & 1 \end{pmatrix} \longrightarrow \left\{ \text{R}_1 \leftarrow \text{R}_1{-}\text{R}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & -1 \\ 5 & 2 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \left\{ \text{R}_2 \leftarrow \text{R}_2 - 5\text{R}_1 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & -10 & 6 \end{pmatrix} \longrightarrow \left\{ \text{R}_2 \leftarrow \frac{1}{2}\text{R}_2 \right\} \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{array} \right)$$

Hence, $A^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$

1.3.2 Column operation

Let $\begin{matrix} \text{C}_1 & \text{C}_2 \\ \downarrow & \downarrow \end{matrix}$ $\left(\begin{array}{cc} 3 & 1 \\ 5 & 2 \\ \hline 1 & 0 \\ 0 & 1 \end{array} \right) = \begin{pmatrix} A \\ \hline E_2 \end{pmatrix}$ be an augmented matrix of $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ for finding A^{-1}.

$$\longrightarrow \left\{ \text{C}_1 \leftrightarrow \text{C}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \left\{ \text{C}_2 \leftarrow \text{C}_2 - 3\text{R}_1 \right\} \longrightarrow \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \\ 1 & -3 \end{pmatrix}$$

$$\longrightarrow \left\{ \text{C}_2 \leftarrow -\text{C}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 3 \end{pmatrix} \longrightarrow \left\{ \text{C}_1 \leftarrow \text{C}_1 - 2\text{R}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \\ -5 & 3 \end{pmatrix} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 2 & -1 \\ -5 & 3 \end{array} \right).$$

Hence, $A^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$

1.4 Example 3 (inverse matrix 3×3)

1.4.1 Row operation

$$\text{Let}\ \begin{matrix} R_1 & \to \\ R_2 & \to \\ R_3 & \to \end{matrix} \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & E_3 \end{array}\right) \text{ be an augmented matrix of } A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

for finding A^{-1}.

$$\longrightarrow \left\{ \mathrm{R_1} \leftrightarrow \mathrm{R_2} \right\} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \left\{ \mathrm{R_3} \leftarrow \mathrm{R_3} - 2\mathrm{R_1} \right\} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -2 & 1 \end{pmatrix}$$

$$\longrightarrow \left\{ \mathrm{R_1} \leftarrow \mathrm{R_1} - \mathrm{R_2} \right\} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -2 & 1 \end{pmatrix} \longrightarrow \left\{ \mathrm{R_3} \leftarrow \mathrm{R_3} + \mathrm{R_2} \right\} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix}$$

$$\longrightarrow \left\{ \mathrm{R_1} \leftarrow \mathrm{R_1} + \mathrm{R_3} \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix} \longrightarrow \left\{ \mathrm{R_2} \leftarrow \mathrm{R_2} - 2\mathrm{R_3} \right\} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\text{Therefore } A^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

1.4.2 Column operation

$$\text{Let}\ \begin{matrix} \mathrm{C_1} & \mathrm{C_2} & \mathrm{C_3} \\ \downarrow & \downarrow & \downarrow \end{matrix} \left(\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) = \left(\frac{A}{E_3}\right) \text{ be an augmented matrix of } A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \text{ for finding } A^{-1}.$$

$$\longrightarrow \left\{ \mathrm{C_1} \leftrightarrow \mathrm{C_2} \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \left\{ \mathrm{C_3} \leftarrow \mathrm{C_3} - 2\mathrm{C_1} \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \left\{ \mathrm{C_1} \leftrightarrow \mathrm{C_1} - \mathrm{C_2} \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \left\{ \mathrm{C_3} \leftarrow \mathrm{C_3} + \mathrm{C_2} \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \left\{ \text{C}_1 \leftrightarrow \text{C}_1\text{+C}_3 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 0 & 1 \end{pmatrix} \longrightarrow \left\{ \text{C}_2 \leftarrow \text{C}_2 - 2\text{C}_3 \right\} \longrightarrow \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & -1 & 1 \\ -1 & 4 & -2 \\ 1 & -2 & 1 \end{array}\right)$$

Therefore $A^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

1.5 Example 4 (Rank)

The **column rank** (resp. **row rank**) of a matrix A with entries in some field is defined to be the maximal number of columns (resp. rows) of A which are linearly independent. The column rank and the row rank are indeed equal; this common number is simply called the **rank of** A. It is commonly denoted by either rk A or rank$A$.

Proposition 1.5.1. rowrank $A =$ columnrank $A =$ rank$A$

$\because$ Let A be an $m \times n$ matrix of columnrank $A = r$, $\{ c_1, c_2, ...c_r \}$ be the linearly independent row vectors, and the $m \times r$ matrix $C = (c_1, c_2, ..., c_r)$. Hence, each column vector of A is a linear combination of the columns of C. Then there exists an $r \times n$ matrix R such that $A = CR$.

Now, from $A = CR$, every row vector of A is a linear combination of the row vectors of R. This means rowrank $A \leqq$ rowrank R. On the other hand, R has r rows, so rowrank $R \leqq r =$ columnrank A. This proves that rowrank $A \leqq$ columnrank A. Now apply the result to the transpose of A to get the reverse inequality columnrank $A =$ rowrank $A^T \leqq$ columnrank $A^T =$ rowrank A. $\therefore$ rowrank $A =$ columnrank A. $\square$

Find the rank of the matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$ using fundamental row operations.

$$\longrightarrow \left\{ \text{R}_1 \leftrightarrow \text{R}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \longrightarrow \left\{ \text{R}_2 \leftarrow \text{R}_2 - 2\text{R}_1 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$$

$$\longrightarrow \left\{ \text{R}_3 \leftarrow \text{R}_3\text{−R}_1 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix} \longrightarrow \left\{ \text{R}_3 \leftarrow \text{R}_3 + 2\text{R}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\longrightarrow \left\{ \text{R}_4 \leftarrow \text{R}_4\text{+R}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{pmatrix} \longrightarrow \left\{ \text{R}_2 \leftrightarrow -\text{R}_2 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\longrightarrow \left\{ \text{R}_4 \leftarrow \frac{1}{4}\text{R}_4 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \left\{ \text{R}_4 \leftarrow \text{R}_4 - \text{R}_3 \right\} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there are 3 nonzero rows remaining in this echelon form of A, $\text{rank}A = 3$

References

[1] Donald J. Wright, Introduction to Linear Algebra, McGraw-Hill, International Editions 1999

[2] Cliffs Notes, Math>Linear Algebra>The Rank of a Matrix, `http://www.cliffsnotes.com/WileyCDA/CliffsReviewTopic/The-Rank-of-a-Matrix.topicArticleId-20807,articleId-20790.html`, (accessed 2012-06-11)