メモ：$L(x) = n$ なる集合

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L を Euler('s torsion) 関数の導来対数関数とし，$M(n) = \{ x \in \mathbb{N} \mid L(x) = n \}$，$m(n) = |M(n)|$, $m(0) = 1$ とする。

さらに，

$M_e(n,\ t) = \{ x : \text{even} \mid x = 2^t x_0,\ x_0 : \text{odd} \quad L(x) = n\}$

$M_e(n) = \{ x : \text{even} \mid L(x) = n \}$

$M_e(n) = \bigcup_{k=1}^{n} M_e(n,\ k),$

$M_p(n) = \{ x : \text{prime} \mid L(x) = n \}$，$c_n = |M_p(n)|$ としておく。

以下簡単に次のように計算できる。

$M_p(1) = \{ 2,\ 3 \}$

$M_p(2) = \{ 5,\ 7 \}$

$M_p(3) = \{ 11,\ 13,\ 19 \}$

$M_p(4) = \{ 17,\ 23, 29,\ 31,\ 37,\ 43 \}$

$M_p(5) = \{ 41,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ 73,\ 79 , 109,\ 127,\ 163\}$

$$|M_e(n)| = \sum_{k=1}^{n} |M_e(n,\ k)|$$

$M_p(6) = \{83, 89, 97, 101, 103, 107, 113, 131, 139, 149,\ \ 151, 157, 173, 181, 191, 197, 199, 211, 223, 229,$
$271, 379, 487\}$

$M_p(7) = \{137, 167, 179, 193, 227, 233, 239, 241, 251, 263,\ \ 269, 277, 281, 283, 293, 307, 311, 313, 317, 331,$
$337, 347, 349, 367, 373, 383, 397, 419, 421, 431,\ \ 433, 439, 457, 463, 491, 509, 523, 541, 547, 571,$
$631, 653, 757, 811, 883, 1459\}$

$M_p(8) = \{257, 353, 359, 389, 401, 409, 443, 449, 461, 467,$
$479, 499, 503, 521, 557, 563, 569, 577, 587, 593,$
$599, 601, 607, 613, 617, 619, 643, 647, 659, 661,$
$673, 677, 683, 691, 701, 709, 727, 733, 739, 743,$
$751, 761, 787, 797, 827, 829, 839, 859, 859, 863,$
$877, 907, 911, 919, 937, 947, 967, 983, 991, 1009,$
$1019, 1033, 1039, 1051, 1063, 1087, 1091, 1093, 1103, 1117,$
$1171, 1279, 1291, 1297, 1303, 1307, 1373, 1423, 1471, 1483,$
$1549, 1567, 1597, 1621, 1627, 1783, 1949, 1999, 2053, 2269,$
$2287, 2647, 2917, 3079 \}$

$M_p(9) = \{$ 641, 719, 769, 773, 809, 821, 823, 857, 881, 887,
929, 941, 953, 971, 977, 997, 1013, 1021, 1031, 1049,
1061, 1069, 1109, 1123, 1129, 1151, 1153, 1163, 1181, 1187,
1193, 1201, 1213, 1217, 1223, 1229, 1231, 1237, 1249, 1259,
1277, 1289, 1301, 1319, 1321, 1327, 1367, 1381, 1399, 1427,
1429, 1447, 1451, 1453, 1481, 1487, 1489, 1493, 1499, 1511,
1523, 1531, 1559, 1671, 1579, 1583, 1697, 1609, 1613, 1657,
1663, 1667, 1669, 1693, 1699, 1709, 1721, 1723, 1733, 1741,
1747, 1753, 1759, 1777, 1787, 1789, 1801, 1811, 1823, 1831,
1847, 1861, 1867, 1873, 1877, 1879, 1901, 1933, 1851, 1979,
1987, 2003, 2011, 2017, 2019, 2039, 2083, 2087, 2089, 2111,
2129, 2131, 2143, 2161, 2179, 2203, 2207, 2213, 2221, 2237,
2239, 2243, 2251, 2281, 2293, 2311, 2341, 2357, 2371, 2377,
2383, 2389, 2399, 2423, 2437, 2503, 2521, 2539, 2549, 2557,
2591, 2593, 2609, 2617, 2677, 2683, 2699, 2711, 2719, 2731,
2749, 2791, 2843, 2851, 2887, 2927, 2971, 3011, 3049, 3067,
3109, 3187, 3253, 3259, 3271, 3307, 3319, 3331, 3511, 3529,
3533, 3547, 3557, 3583, 3613, 3727, 3823, 3889, 3907, 3919,
3943, 4051, 4159, 4219, 4447, 4549, 4663, 4789, 4861, 4871,
4903, 5023, 5347, 5419, 5869, 6823, 6967, 8803 $\}$

$M_p(10) = \{$ 1097, 1283, 1361, 1409, 1433, 1439, 1543, 1553, 1601, 1619,
1637, 1697, 1871, 1889, 1907, 1913, 1931, 1973, 1993, 1997,
2027, 2063, 2069, 2081, 2099, 2113, 2137, 2141, 2153, 2257,
2273, 2297, 2309, 2333, 2339, 2347, 2351, 2381, 2393, 2411,
2417, 2441, 2447, 2459, 2467, 2473, 2477, 2531, 2543, 2551,
2579, 2621, 2633, 2659, 2663, 2671, 2687, 2689, 2693, 2707,
2713, 2729, 2753, 2767, 2777, 2797, 2801, 2803, 2833, 2837,
2857, 2861, 2897, 2903, 2909, 2939, 2953, 2957, 2963, 2969,
2999, 3001, 3019, 3023, 3037, 3041, 3061, 3083, 3119, 3121,
3137, 3163, 3167, 3169, 3181, 3191, to be continued.... $\}$

また，次の恒等式は明らかであろう。

$$\sum_{k=0}^{\infty} m(k)x^k = \prod_{k=1}^{\infty}\prod_{i=1}^{c_k}\frac{1}{1-x^k} = \prod_{k=1}^{\infty}\frac{1}{(1-x^k)^{c_k}}$$

$m(n)$ までを求めたければ

$$\prod_{k=1}^{n} \frac{1}{(1-x^k)^{c_k}}$$

の計算でよい。

例えば，$c_1 = 2,\ c_2 = 2,\ c_3 = 3,\ c_4 = 6,\ c_5 = 12,\ c_6 = 23,\ c_7 = 46,\ c_8 = 94,\ c_9 = 198, \ldots$ より

$$\begin{aligned}\prod_{k=1}^{9} \frac{1}{(1-x^k)^{c_k}} &= \frac{1}{(1-x)^2}\cdot\frac{1}{(1-x^2)^2}\cdot\frac{1}{(1-x^3)^3}\cdot\frac{1}{(1-x^3)^3}\cdot\frac{1}{(1-x^4)^6}\cdot\\ &\quad \frac{1}{(1-x^5)^{12}}\cdot\frac{1}{(1-x^6)^{23}}\cdot\frac{1}{(1-x^7)^{46}}\cdot\frac{1}{(1-x^8)^{94}}\cdot\frac{1}{(1-x^9)^{198}}\\ &= 1 + 2x + 5x^2 + 11x^3 + 26x^4 + 59x^5 + 137x^6 + 312x^7 + 719x^8 + 1649x^9 + \cdots\end{aligned}$$

$\therefore\quad m(1) = 2,\ m(2) = 5,\ m(3) = 11,\ m(4) = 26,\ m(5) = 59,\ m(6) = 137,\ m(7) = 312,\ m(8) = 719,\ m(9) = 1649,\ \ldots$

また

$$\max_{q\in M_p(n)} q \leqq 2\cdot 3^{n-1} + 1$$

上記の等号は外せない。$2\cdot 3^{n-1}+1$ が素数になる場合があることより。ただし，$n \longrightarrow \infty$ では，（実験では）素数になる確率は下がってゆくような振舞いである・・・。

因みに $n < 301$ の範囲で $2{\cdot}3^{n-1}{+}1$ が素数となるのは $n = 2, 3, 6, 7, 10, 17, 18, 31, 55, 58, 61, 66, 133, 181$ の 14 個だけである（by Maple）。

$2\cdot 3^{181-1} + 1 =$

1523546 9609173278 4678579455 4412311235 0084960280 4790393448 0031314899
1427468606 6076039203