Calculation Memo : 20240330
On convolutional power of Euler function φ

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Convolutinal power of Arithmetic function

Define the convolutinal $k-$th power $f^{(k)}$ of arithmetic function f as follows

$$f^{(2)}(n) = (f * f)(n)$$
$$f^{(k)}(n) = (f * f^{(k-1)})(n) \quad k \geqq 3$$

Convolutinal power of Euler function φ

Let n be considered below as a power of a prime number p.

$$\varphi^{(2)}(1) = \varphi(1) \cdot \varphi(1) = 1$$
$$\varphi^{(2)}(p) = \varphi(1) \cdot \varphi(p) + \varphi(p) \cdot \varphi(1) = (p-1) + (p-1) = 2(p-1)$$
$$\varphi^{(2)}(p^2) = \varphi(1) \cdot \varphi(p^2) + \varphi(p) \cdot \varphi(p) + \varphi(p^2) \cdot \varphi(1) = p(p-1) + (p-1)^2 + p(p-1)$$
$$= 3(p-1)(2p-1)$$
$$\varphi^{(2)}(p^3) = \varphi(1) \cdot \varphi(p^3) + \varphi(p) \cdot \varphi(p^2) + \varphi(p^2) \cdot \varphi(p) + \varphi(p^3) \cdot \varphi(1)$$
$$= p^2(p-1) + (p-1) \cdot p(p-1) + p(p-1) \cdot (p-1) + p^2(p-1) = p(p-1)(4p-2)$$
$$\varphi^{(2)}(p^4) = \varphi(1) \cdot \varphi(p^4) + \varphi(p) \cdot \varphi(p^3) + \varphi(p^2) \cdot \varphi(p^2) + \varphi(p^3) \cdot \varphi(p) + \varphi(p^4) \cdot \varphi(1)$$
$$= p^3(p-1) + (p-1) \cdot p^2(p-1) + p(p-1) \cdot p(p-1) + p^2(p-1) \cdot (p-1) + p^3(p-1)$$
$$= p^2(p-1)(5p-3)$$
$$\cdots\cdots$$
$$\varphi^{(2)}(p^e) = p^{e-2}(p-1)\ (\ (e+1)p - (e-1)\ ) = p^{e-2}(p-1)\ (\ (e+1)(p-1) + 2\ )$$

In case of $\varphi^{(3)}$

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\begin{aligned}
\varphi^{(3)}(1) &= \varphi(1)\cdot\varphi^{(2)}(1) = 1\\
\varphi^{(3)}(p) &= \varphi(1)\varphi^{(2)}(p) + \varphi(p)\varphi^{(2)}(1) = 2(p-1) + (p-1) = 3(p-1)\\
\varphi^{(3)}(p^2) &= \varphi(1)\varphi^{(2)}(p^2) + \varphi(p)\varphi^{(2)}(p) + \varphi(p^2)\varphi^{(2)}(1)\\
&= (p-1)(3p-1) + (p-1)\cdot 2(p-1) + p(p-1) = 3(p-1)(2p-1)\\
\varphi^{(3)}(p^3) &= \varphi(1)\varphi^{(2)}(p^3) + \varphi(p)\varphi^{(2)}(p^2) + \varphi(p^2)\varphi^{(2)}(p) + \varphi(p^3)\varphi^{(2)}(1)\\
&= p(p-1)(4p-2) + (p-1)\cdot(p-1)(4p-2) + p(p-1)\cdot 2(p-1) + p^2(p-1)\\
&=
\end{aligned}
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References

[1] M. Yamashita, Computational Experimental Memorandum:On the iteration of basic arithmetic functions,
http://yamashita-lab.net/yamasita-diary/arithm20240303.pdf (Reference date 2024/3/3)