On the calculation memo for 20230804 : Various properties of ψ_0 functions

Michinori Yamashita* Daisuke Miyata† Natsumi Fujita‡

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Let us slightly modify Dedekind's ψ function to define the following function ψ_0.

Definition 1.

$$\psi_0(n) = \begin{cases} 1 & (\ n = 1\ ) \\ \prod_i p_i^{e_i - 1} \left\lfloor \frac{p_i + 1}{2} \right\rfloor & (\ n > 1\ ) \end{cases} \quad , \textit{where} \ \ n = \prod_i p_i^{e_i}$$

Convolution of ψ_0 and Möbius function μ, $\psi_0 * \mu(n) = \sum_{d|n} \psi_0(d) \mu\left(\frac{n}{d}\right)$.

Let's calculate some of them.

When $n = p$ prime

$$\psi_0 * \mu(p) = \psi_0(p) - \psi_0(1) = \begin{cases} 0 & (\ p = 2\ ) \\ \frac{p-1}{2} = \frac{\varphi(p)}{2} & (\ p > 2\ ) \end{cases}$$

When $n = 2^e,\ e > 1$

$$\psi_0 * \mu(2^e) = \psi_0(2^e) - \psi_0(2^{e-1}) = 2^{e-1} - 2^{e-2} = 2^{e-2} = \frac{\varphi(2^e)}{2}$$

* Rissho university
† Chiba university of Commerce
‡ Fujitsu Japan Limited

When $n = 2p$ $(p : \text{prime}, p > 2\ )$

$$\begin{aligned}\psi_0 * \mu(2p) &= \psi_0(1)\mu(2p) + \psi_0(2)\mu(p) + \psi_0(p)\mu(2) + \psi_0(2p)\mu(1) \\ &= 1 \cdot 1 + 1 \cdot (-1) + \frac{p+1}{2} \cdot (-1) + \frac{p+1}{2} \cdot 1 = 0\end{aligned}$$

When $n = 2p^e$ $(p : \text{prime}, p > 2,\ e > 1\ )$

$$\begin{aligned}\psi_0 * \mu(2p^e) &= \psi_0(2p^e)\mu(1) + \psi_0(p^e)\mu(2) + \psi_0(2p^{e-1})\mu(p) + \psi_0(p^{e-1})\mu(2p) \\ &= p^{e-1}\frac{p+1}{2} \cdot 1 + p^{e-1}\frac{p+1}{2} \cdot (-1) + p^{e-2}\frac{p+1}{2} \cdot (-1) + p^{e-2}\frac{p+1}{2} \cdot 1 \\ &= 0\end{aligned}$$

When $n = 2^g p^e$ $(p : \text{prime}, p > 2,\ g,\ e > 1\ )$

$$\begin{aligned}\psi_0 * \mu(2^g p^e) &= \psi_0(2^g p^e)\mu(1) + \psi_0(2^{g-1}p^e)\mu(2) + \psi_0(2^g p^{e-1})\mu(p) + \psi_0(2^{g-1}p^{e-1})\mu(2p) \\ &= 2^{g-1}p^{e-1}\frac{p+1}{2} \cdot 1 + 2^{g-2}p^{e-1}\frac{p+1}{2} \cdot (-1) + 2^{g-1}p^{e-2}\frac{p+1}{2} \cdot (-1) \\ &\quad + 2^{g-2}p^{e-2}\frac{p+1}{2} \cdot 1 \\ &= 2^{g-2}p^{e-2}\frac{p+1}{2}(p-1) = \psi_0(2p)\varphi(2^{g-1}p^{e-1}) = \psi_0(2p)\frac{\varphi(2^g p^{e-1})}{2}\end{aligned}$$

When $n = p^e q^f$ $(p,\ q : prime, p,\ q > 2,\ e,\ f > 1\ )$

$$\begin{aligned}\psi_0 * \mu(p^e q^f) &= \psi_0(p^e q^f)\mu(1) + \psi_0(p^{e-1}q^f)\mu(p) + \psi_0(p^e q^{f-1})\mu(q) + \psi_0(p^{e-1}q^{f-1})\mu(pq) \\ &= p^{e-1}q^{f-1}\frac{p+1}{2}\frac{q+1}{2} - p^{e-2}q^{f-1}\frac{p+1}{2}\frac{q+1}{2} \\ &\quad - p^{e-1}q^{f-2}\frac{p+1}{2}\frac{q+1}{2} + p^{e-2}q^{f-2}\frac{p+1}{2}\frac{q+1}{2} \\ &= \frac{p+1}{2}\frac{q+1}{2}p^{e-2}q^{f-2}(pq - p - q + 1) = \psi_0(pq)\varphi(p^{e-1}q^{f-1}) \\ &= \psi_0(\text{rad }(n))\varphi\left(\frac{n}{\text{rad }(n)}\right)\end{aligned}$$

When $n = 2p^e q^f$ $(p,\ q : prime, p,\ q > 2,\ e,\ f > 1\ )$

$$\begin{aligned}
\psi_0 * \mu(2p^e q^f) &= \psi_0(2p^e q^f)\mu(1) \\
&\quad + \psi_0(p^e q^f)\mu(2) + \psi_0(2p^{e-1} q^f)\mu(p) + \psi_0(2p^e q^{f-1})\mu(q) \\
&\quad + \psi_0(p^{e-1} q^f)\mu(2p) + \psi_0(p^e q^{f-1})\mu(2q) + \psi_0(2p^{e-1} q^{f-1})\mu(pq) \\
&\quad + \psi_0(p^{e-1} q^{f-1})\mu(2pq) \\
&= \psi_0(2p^e q^f) \cdot 1 \\
&\quad + \psi_0(p^e q^f)(-1) + \psi_0(2p^{e-1} q^f)(-1) + \psi_0(2p^e q^{f-1})(-1) \\
&\quad + \psi_0(p^{e-1} q^f) \cdot 1 + \psi_0(p^e q^{f-1}) \cdot 1 + \psi_0(2p^{e-1} q^{f-1}) \cdot 1 \\
&\quad + \psi_0(p^{e-1} q^{f-1})(-1) \\
&= p^{e-1} q^{f-1} \frac{p+1}{2} \frac{q+1}{2} \\
&\quad - p^{e-1} q^{f-1} \frac{p+1}{2} \frac{q+1}{2} - p^{e-2} q^{f-1} \frac{p+1}{2} \frac{q+1}{2} - p^{e-1} q^{f-2} \frac{p+1}{2} \frac{q+1}{2} \\
&\quad + p^{e-2} q^{f-2} \frac{p+1}{2} \frac{q+1}{2} \\
&= \frac{p+1}{2} \frac{q+1}{2} p^{e-2} q^{f-2} (p-1)(q-1) \\
&= \psi_0(pq)\varphi(p^{e-1} q^{f-1})
\end{aligned}$$

From the above calculations, the form of $\psi_0 * \mu(n)$ is determined for general n.

$$\psi_0 * \mu(n) = \begin{cases} \dfrac{\varphi(n)}{2} & (\ n : \text{ odd prime }) \\ 0 & (\ 2 || n,\ \omega(n) \leqq 2\ ) \\ \psi_0(\text{rad }(n))\varphi\left(\dfrac{n}{\text{rad }(n)}\right) & (\ \text{otherwise}\ ) \end{cases}$$

where $\omega(n)$ is the number of distinct prime factors of n.

References

[1] M.Yamashita, D.Miyata, K.Shibaki: Note on the derived logarithmic fuction of a multiplicative arithmetic function, Proceedings of 1^{st}IAR2023, IIARS, Fukui, 2023.9.8−9.11 (in preparation).