Note on modified Dedekind's $\psi-$function

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1 Introduction

Dedekind's ψ function is given by

$$\psi(n) = n \prod_{\substack{p|n \\ p:\text{prime}}} \left(1 + \frac{1}{p}\right) = \prod_{\substack{i=1 \\ p_i|n}}^{\omega(n)} p_i^{e_i-1}(p_i + 1)$$

Here let's ψ_0 function be the following arithmetic function ψ_0 defined for primes p or 1 such as

$$\psi_0(p) = \begin{cases} 1 & (p = 1) \\ 1 & (p = 2) \\ \dfrac{p+1}{2} & (p > 2) \end{cases}$$

Then

$$\psi_0(n) = \prod_{\substack{p|n \\ p:\text{prime}}}^{\omega(n)} p^{e-1}\psi_0(p) = \frac{1}{2^{\omega(n)}} \cdot n \prod_{\substack{p|n \\ p:\text{prime}}}^{\omega(n)} \left(1 + \frac{1}{p}\right)$$

where $\omega(n)$ is the number of discinct prime factors of n

ψ of primes

p	ψ	p	ψ	p	ψ	p	ψ	p	ψ	p	ψ	p	ψ	p	ψ	p	ψ
2	1	13	7	31	16	53	27	73	37	101	51	127	64	151	76	179	90
3	2	17	9	37	19	59	30	79	40	103	52	131	66	157	79	181	92
5	3	19	10	41	21	61	31	83	42	107	54	137	69	163	82	191	96
7	4	23	11	43	22	67	34	89	45	109	55	139	70	167	84	193	97
11	6	29	15	47	24	71	36	97	49	113	57	149	75	173	87	197	99

Proposition 1.1. (MIYATA − YAMASHITA) Let $\mathcal{P}$ be the set of all primes $f : \mathcal{P} \to \mathcal{N}$ be a function such as $1 \leqq f(p) < p \in \mathcal{P}$

$$\varphi_f(x) = x \prod_{i=1}^{r} \frac{f(p_i)}{p_i}$$

and

$$L_{\varphi_f}(1) = 0$$
$$L_{\varphi_f}(x) = L_{\varphi_f}(\varphi_f(x)) + \#\{\ p \in f^{-1}(1) :\ p|x\ \}.$$

Then

$$L_{\varphi_f}(xy) = L_{\varphi_f}(x) + L_{\varphi_f}(y)$$

Hence, we apply ψ_0 to f in Proposition1.1, for modified Dedekind function ψ, we get

$$L_{\varphi_{\psi_0}}(xy) = L_{\varphi_{\psi_0}}(x) + L_{\varphi_{\psi_0}}(y)$$

i.e., for the modified Dedekind's arithmetic function ψ_0 we get a relation of tyhe derived logarithmic function.

References

[1] M.Yamashita, D.Miyata: On the *abc* conjecture for a derived logarithmic function of the Euler function, Proceedings of 1th CCATS2015_IEEE(International Conference on Computer Application & TechnologieS 2015), Session #7(9.2), Kunibiki Messe(Matsue), 2015.8.31−9.2

[2] D.Miyata-M.Yamashita, Note on derived logarithmic functions of Euler's functions, Proceedings of Autum meeting(App. Math.), Math. Soc. of Japan, 2004.9, (in Japanese)

[3] Miyata D.-Yamashita M., A generalization of Yamashita's note on derived logarithmic functions of Euler functions,2001.12.14 (unpublished)

[4] Dedekind psi function: n * Product$_{\{\ p|n, p\text{prime}\ \}}(1 + 1/p)$.
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