# Sophia University

# Applied Mathematics II

# Final Examination

Examiner: Professor M. Yamashita | Date:Monday, January 22, 2007

Time 15:30 P.M. – 16:15 P.M.

Read the directions to each problem carefully. Show your work and answer in complete sentences when appropriate. This exam has 12 questions on 4 pages including this cover.

**Problem 0.** (20 points each) Explain the following mathematical approches to the things.

(1) linear / non-linear

(2) local / global or micro / macro

(3) deterministic / stochastic

(4) static / dynamic

(5) continuous / discrete

(6) general situation / special situation

(7) fractal / chaos

**Problem 1.** (12 points each) Carefully state the followings.

(1) piecewise continuous

(2) piecewise smooth

(3) Fourier, Jean-Baptiste-Joseph

(4) convolutions

(5) Fourier coefficients of $f(x)$

(6) Fourier series of complex form

(7) Parseval's equality

(8) convolutions

(9) the convolution theorem of Fourier transforms

(10) Fourier transforms

(11) Friedrich Wilhelm Bessel

(12) partial differntial equation

(13) Gibbs' phenomenon

(14) generating function

(15) Fourier polynomial

**Problem 2.** (8 points each) Find the following definite integrals where the integers $m,\ n \geq 0$.

(1) $\displaystyle\int_0^{\pi} \frac{1+\cos x}{x+\sin x}\, dx$

(2) $\displaystyle\int_0^{1} x \log x \; dx$

(3) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx \; dx$

(4) $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx \; dx$

(5) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \; dx$

(6) $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x} \; dx$

**Problem 3.** (15 points each) Suppose that $f(x)$ is a continuous function over the interval $[-\pi, \pi]$. Show the following equalities.

(1) $\displaystyle\int_{0}^{\pi/2} f(\sin x) \; dx = \int_{0}^{\pi/2} f(\cos x) \; dx$

(2) $\displaystyle\int_{0}^{\pi} f(\sin x) \; dx = 2\int_{0}^{\pi/2} f(\sin x) \; dx$

(3) $\displaystyle\int_{0}^{\pi} x f(\sin x) \; dx = \frac{\pi}{2}\int_{0}^{\pi} f(\sin x) \; dx$

**Problem 4.** (20 points) Show the work necessary to verify that for arbitrary $C^2$ functions $f$ and $g$, the expression

$$u = f(x + ct) - g(x - ct)$$

is a solution of the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

where $c$ is a positive constant.

**Problem 5.** (30 points) Let $S_n(x)$ denote the $n$-th order partial sum of the Fourier series for a function $f$. Show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} S_n(x)^2 \; dx = \frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)$$

where the $a_k$'s and $b_k$'s are the Fourier coefficients of $f$.

**Problem 6.** (20 points) Find the Fourier coefficients of the square wave function

$$w(x) = \begin{cases} a & 0 \le x \le \pi, a > 0, \\ -b & -\pi < x < 0, b > 0. \end{cases}$$

and write down the first four terms of the Fourier series.

**Problem 7.** (20 points) The Heaviside function $\mathcal{H}(x)$ is given by

$$\mathcal{H}(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geqq 0 \end{cases}$$

Find the Fourier transform of $\mathcal{H}(x)e^{-ax}$.

**Problem 8.** (40 points) Define $f(x)$ as follows:

$$f(x) = \begin{cases} 3x & 0 < x < 1 \\ 0 & \text{othewise} \end{cases}$$

(1) Find the Fourier transform of $f$.

(2) Express $f(x)$ as a Fourier integral.

(3) Express $f(x)$ as a real Fourier integral.

**Problem 9.** (30 points) Use a Fourier transform to solve the heat conduction problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad x \geqq 0\ ,$$

$$u(x,0) = x \exp\left(-\frac{\mathrm{x}^2}{4\mathrm{a}^2}\right) \qquad ,$$

$$u(0,t) = 0$$

**Problem 10.** (30 points) By taking the Fourier transform of

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x > 0. \end{cases}$$

find a formula for integrals of the form

$$\int_0^\infty \frac{\xi \sin(\xi x) + \cos(\xi x)}{1+\xi^2}\, d\xi$$

**Problem 11.** (50 points) Explain the useful techniques for solving PDEs(Partial Differntial Equations).