Environmental Info-Mathematics II [1H105602]

Final Examination

Examiner: YAMASHITA M.

Date: Wednesday, July 18, 2012

Time: 09:00 A.M. – 10:10 A.M.

Read the directions to each problem carefully. Show your work and answer in complete sentences when appropriate. This exam has 16 questions on 4 pages including this cover.

Problem 0. (5 points each) Explain the definitins of the followings. It does not mean to translate the followings into japanese mathematical term.

(1) matrix
(2) trace
(3) transposed matrix
(4) symmetric matrix
(5) Gaussian Elimination
(6) elementary operation (fundamental operation)
(7) linear combination
(8) dot product
(9) cross product
(10) determinant
(11) minor
(12) cofactor
(13) cofactor expansion (Laplace expansion)
(14) minimal polynomial
(15) eigenvalue
(16) eigenvector
(17) diagonalizable

Problem 1. (15 points) Solve the following linear system using the fundamental operations.

$$\begin{cases} x+y+2z &= 9 \\ 2x+y-z &= 1 \\ -x+2y+2z &= 9 \end{cases}$$

Problem 2. (25 points) Find the rank of a matrix A using the fundamental operations.

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

Problem 3. (25 points) Find the inverse of a matrix A using the fundamental operations.

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -5 & 0 & 3 \end{pmatrix}$$

Problem 4. (15 points) Find the inverse of the reflection matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Problem 5. (20 points) Find a diagonlizable matrix A given the eigenvalues and associated eigenvectors of A.

$$\lambda = -1, \begin{pmatrix} 2 \\ -5 \end{pmatrix} \qquad \lambda = 2, \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Problem 6. (30 points) State and prove the Cayley-Hamilton Theorem.

Problem 7. (35 points) Show that if A is an $n \times n$ matrix, then A^n can be written as a linear combination of the matrices $I, A, A^2, ..., A^{n-1}$ (that is, $A^n = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_{n-1} A^{n-1}$ for some scalars $\alpha_0, ..., \alpha_{n-1}$ where I is the $n \times n$ identity matrix.

Problem 8. (each 10 points) Let $\mathrm{i} = (1, 0, 0),\ \mathrm{j} = (0, 1, 0),\ \mathrm{k} = (0, 0, 1)$. Evaluate each expression.

(1) $(\mathrm{i} + \mathrm{j}) \times (\mathrm{i} - \mathrm{j} + \mathrm{k}) \cdot (\mathrm{i} + \mathrm{j} + \mathrm{k})$

(2) $(\mathrm{i} - 2\mathrm{j} + \mathrm{k}) \times (\mathrm{i} + \mathrm{k}) \times (\mathrm{j} + \mathrm{k})$

Problem 9. (15 points each) Let $\mathrm{u} = (-1, 2, 3),\ \mathrm{v} = (1, -2, 3),\ \mathrm{w} = (0, 1, -1)$. Compute each expression.

(1) $\mathrm{u} \times \mathrm{v}$

(2) $\mathrm{u} \cdot (\mathrm{v} \times \mathrm{w})$

(3) $\mathrm{u} \times (\mathrm{v} \times \mathrm{w})$

Problem 10. (15 points each) Evaluate the following determinants.

(1) $\begin{vmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{vmatrix}$ (2) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ (3) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & a & a \\ x & y & b & b \\ x & y & z & c \end{vmatrix}$

(4) $\begin{vmatrix} 1+a & 1 \\ 1 & 1+a \end{vmatrix}$ (5) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$

Problem 11. (15 points each) Find the eigenvalues and an associated eigenvector for each eigenvalue.

$$(1)\ \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \qquad (2)\ \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad (3)\ \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}$$

Problem 12. (20 points each) Determine if the given matrix A is diagonalizable. If so, find a diagonalizing matrix P for A and check that $P^{-1}AP = D$.

$$(1)\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad (2)\ \begin{pmatrix} 15 & 6 \\ -6 & 3 \end{pmatrix} \qquad (3)\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 13. (20 points) If $(a-d)^2 + 4bc > 0$, show that the real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is diagonalizable.

Problem 14. (30 points) A matrix A has characteristic polynomial $f_A(x) = (x-1)^3$ and minimal polynomial $m_A(x) = (x-1)^2$. List all possible Jordan canonical forms for A.

Problem 15. (15 points each) Let $A = \begin{pmatrix} -2 & -6 & 10 \\ 6 & 11 & 15 \\ -2 & -3 & -3 \end{pmatrix}$.

(1) Find the characteristic polynomial of A.

(2) Find the minimal polynomial of A.

(3) Find the eigenvalues of A.

(4) Find the Jordan (canonical) form of A.