On the *abc* conjecture for a derived logarithmic fuction of the Euler function

Michinori Yamashita
Faculty of Geo-Environmental Science
Rissho University
Saitama, 360-0194 JAPAN
Email: yamasita@ris.ac.jp

Daisuke Miyata
Faculty of Commerce and Economics
Chiba University of Commerce
Chiba, 272-0827 JAPAN
Email: miyata@cuc.ac.jp

***Abstract*—At first we review the properties of the derive logarithmic function L, and we state the *abc* conjecture for L of Euler function φ. Thus we proved the *abc* conjecture for L holds for some cases.**

I. Introduction

The study of the arithmetic function arising of the Euler $\varphi(n)$ function, representing the number of natural numbers relatively prime to given, n has been done by some researchers, for example by Pillai [7][8], Shapiro [6] and Murányi [5]. Let $\varphi_n(x)$ be the iterated φ function such that $\varphi_1(x) = \varphi(x)$, $\varphi_n(x) = \varphi(\varphi_{n-1}(x))$, and let $C(x)$ denote the smallest k satisfying the condition $\varphi_k(x) = 2$ for $x \geq 3$ and $C(1) = C(2) = 0$.

Theorem 1: (Shapiro [6], [4], Murányi [5]) For two integers x and y, if either x or y is odd

$$C(xy) = C(x) + C(y),$$

and if both x and y are even

$$C(xy) = C(x) + C(y) + 1.$$

In 1971, one of authors has introduced the notion of the derived logarithmic function L of Euler function φ.

Definition 2: (Yamashita [1]) The "*derived logarithmic function L*" of Euler function φ is defined by

$$L(x) = \begin{cases} L(\varphi(x)) & x \text{ is odd} \\ L(\varphi(x)) + 1 & x \text{ is even} \end{cases},$$

where $L(1) = 0$.

Since the Euler function φ is monotonically decreasing function of n, the function with recursive definition in the formula above is well-defined.

Then, showed the next property for L.

Proposition 3: (Yamshita) For $x, y \in \mathcal{N}$, the set of natural numbers,

$$L(xy) = L(x) + L(y)$$

We extended the range of the function L from $\mathcal{N}$ to $\mathcal{Q}$, where $\mathcal{Q}$ is the set of rational numbers, by

$$L\left(\frac{x}{y}\right) = L(x) - L(y), \quad \text{where } \frac{x}{y} \in \mathcal{Q}$$

.

II. The properties of derived logarithmic function L_f of Euler function φ_f

In 2001, we also generalized *Propotion* 3.

Proposition 4: (Miyata, Yamashita [3]) Let P be the set of prime numbers and $f : \mathbf{P} \to \mathcal{N}$ be a function such that $1 \leqq f(p) < p$ and we define

$$\varphi_f(x) = x \prod_{i=1}^{r} \frac{f(p_i)}{p_i},$$

where $x = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, and

$$\begin{aligned} L_f(1) &= 0 \\ L_f(x) &= L(\varphi_f(x)) + \#\{p \in f^{-1}(1) : p|x\}. \end{aligned}$$

Then, we have

$$L_f(xy) = L_f(x) + L_f(y).$$

Proof.

Using an induction argument on $x = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, it is sufficient to show

$$L_f(x) = e_1 L_f(p_1) + e_2 L_f(p_2) + \cdots + e_r L_f(p_r)$$

[1]Private Communication between Yamashita and Tsukuba Univ. Prof. Dr. Uchiyama S. :On a derived logarithmic function of an Euler's function, 1977.9.10. Yamashita knew the works of Pillai and Shapiro for the fiorst time.

$\alpha := \#\{p \in f^{-1}(1) : p|x\} \cdot \mathrm{C}$

$$\begin{aligned} L_f(x) &= L_f(\varphi(x)) + \alpha \\ &= \sum_{i=1}^{r}(e_i - 1)L_f(p_i) + \sum_{i=1}^{r} L_f(f(p_i)) + \alpha \\ &= \sum_{i=1}^{r}(e_i - 1)L_f(p_i) + \sum_{i=1}^{r} L_f(p_i) - \alpha + \alpha \\ &= \sum_{i=1}^{r} e_i L_f(p_i) \end{aligned}$$

□

We notice that if $f(p) = 1$ then $L_f(f(p)) = 0$, else $L_f(f(p)) = L_f(p)$ for $p \in \mathbf{P}$.

III. The *abc* conjecture for L

Let $\mathrm{rad}(x)$ be a radical of x, that is if $x = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ then $rad(x) = p_1 p_2 \cdots p_r$. Using the computer resources, we verified the followings,

Proposition 5: Let $a, b, c \in \mathcal{N}$ be relatively prime numbers satisfying $a + b = c$. If $c \leqq 10^5$, then

$$\max(L(a), L(b), L(c)) \leqq 2 \cdot L(\mathrm{rad}(abc))$$

The following is an immediate result to 2 and show a property of L.

Lemma 6: For $x \in \mathcal{N}$,

1) if $L(x) \geqq n$, then $x \geqq 2^n$
2) if $L(x) \leqq n$, then $x \leqq 3^n$

Corollary 7:

1) if $x \leqq 2^n$, then $L(x) \leqq n$
2) if $x \geqq 3^n$, then $L(x) \geqq n$

Lemma 8: For all natural numbers x, y satisfying $\max(L(x), L(y)) \geqq 3$,

$$L(x + y) \leqq 2 \cdot \max(L(x), L(y))$$

Proof.

By *Lemma* 6, $x \leqq 3^{L(x)}$ and $y \leqq 3^{L(y)}$,

$$\begin{aligned} x + y &\leqq 3^{L(x)} + 3^{L(y)} \leqq 2 \cdot 3^{\max(L(x),L(y))} \\ &\leqq 2^{2\max(L(x),L(y))}. \end{aligned} \tag{1}$$

Thus, by *Corollary* 7,

$$L(x + y) \leqq 2 \cdot \max(L(x), L(y))$$

□

The next *Lemma* is proved in the same way as discussed in the proof of *Lemma* 8.

Lemma 9: For all natural numbers $x > y$,

$$L(x - y) \leqq 2L(x)$$

Corollary 10: For all natural numbers x, y, z, x', y', z', If $x + y - z > 0$

$$L(x + y - z) \leqq 2 \cdot \max(L(x), L(y)).$$

If $x' - y' - z' > 0$

$$L(x' - y' - z') \leqq 2L(x')$$

Proposition 11: Let $a, b, c \in \mathcal{N}$ be relatively prime numbers satisfying $a + b = c$. If $\varphi(a) + \varphi(b) = \varphi(c)$, then

$$\max(L(a), L(b), L(c)) \leqq 2L(\mathrm{rad}(abc))$$

Proof.

From $a + b = c$ and $\varphi(a) + \varphi(b) = \varphi(c)$,

$$a\varphi(c) - c\varphi(a) + b\varphi(c) - c\varphi(b) = 0,$$

and

$$ac\left(\frac{\varphi(c)}{c} - \frac{\varphi(a)}{a}\right) = bc\left(\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}\right).$$

If

$$\frac{\varphi(b)}{b} = \frac{\varphi(c)}{c},$$

then it contradicts to the assumption that b and c is relatively prime. Thus we have that

$$\frac{\varphi(b)}{b} \neq \frac{\varphi(c)}{c},$$

and

$$\frac{\varphi(c)}{c} \neq \frac{\varphi(a)}{a}.$$

Therefore

$$\begin{aligned} \frac{a}{b} &= \frac{\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}}{\frac{\varphi(c)}{c} - \frac{\varphi(a)}{a}} \\ &= \frac{\mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}\right)}{\mathrm{rad}(abc)\left(\frac{\varphi(c)}{c} - \frac{\varphi(a)}{a}\right)} \end{aligned}$$

Here, we notice that

$$\mathrm{rad}(abc) \cdot \frac{\varphi(a)}{a},\ \mathrm{rad}(abc) \cdot \frac{\varphi(b)}{b},\ \mathrm{rad}(abc) \cdot \frac{\varphi(c)}{c} \in \mathcal{N}$$

and

$$L\left(\frac{\varphi(a)}{a}\right),\ L\left(\frac{\varphi(b)}{b}\right), L\left(\frac{\varphi(c)}{c}\right) = 0 \text{ or } 1.$$

Since $\frac{a}{b}$ is ireducible,

$$\begin{aligned} a \,\Big|\, & \mathrm{rad}(abc)\left(\frac{\varphi(c)}{c} - \frac{\varphi(a)}{a}\right) \\ &= \mathrm{rad}(abc)\frac{\varphi(c)}{c} - \mathrm{rad}(abc)\frac{\varphi(a)}{a}. \end{aligned} \tag{2}$$

Therefore, by Lemma8, we see that

$$L(a)2 \leqq L(\mathrm{rad}(abc)).$$

Similarly,

$$L(b) \leqq 2L(\mathrm{rad}(abc)).$$

Moreover, we notice that

$$\frac{c}{b} = \frac{\frac{\varphi(a)}{a} - \frac{\varphi(b)}{b}}{\frac{\varphi(a)}{a} - \frac{\varphi(c)}{c}} = \frac{\mathrm{rad}(abc)\left(\frac{\varphi(a)}{a} - \frac{\varphi(b)}{b}\right)}{\mathrm{rad}(abc)\left(\frac{\varphi(a)}{a} - \frac{\varphi(c)}{c}\right)}.$$

Finally we have

$$L(c) \leqq 2L(\mathrm{rad}(abc)).$$

□

Proposition 12: Let $a, b, c \in \mathcal{N}$ be relatively prime numbers satisfying $a + b = c$ and $\varphi(a) + \varphi(b) < \varphi(c)$. If $a = \mathrm{rad}(a)$ then

$$\max(L(a), L(b), L(c)) \leqq 2 \cdot L(\mathrm{rad}(abc))$$

Proof.

We may assume that $L(a), L(b), L(c) \geqq 3$ by Proposition 9 and that $\varphi(a) + \varphi(b) + d = \varphi(c)$, where $d > 0$.

By the proof of *Proposition* 11, we see that

$$\frac{b}{a} = \frac{\frac{\varphi(c)}{c} - \frac{\varphi(a)}{a} - \frac{d}{a}}{\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}} = \frac{\mathrm{rad}(abc)\left(\frac{\varphi(c)}{c} - \frac{\varphi(a)}{a}\right) - \mathrm{rad}(abc)\frac{d}{a}}{\mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}\right)},$$

$$\frac{c}{a} = \frac{\frac{\varphi(b)}{b} - \frac{\varphi(a)}{a} \mp \frac{d}{a}}{\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}} = \frac{\mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(a)}{a}\right) - \mathrm{rad}(abc)\frac{d}{a}}{\mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}\right)}.$$

Hence, since $\mathrm{rad}(a) = a$

$$\mathrm{rad}(abc)\frac{d}{a} \in \mathcal{N}.$$

Moreover, since $\frac{b}{a}$ and $\frac{c}{a}$ are irreducible,

$$a \,\Big|\, \mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}\right) = \mathrm{rad}(abc)\frac{\varphi(b)}{b} - \mathrm{rad}(abc)\frac{\varphi(c)}{c},$$

$$b \,\Big|\, \mathrm{rad}(abc)\left(\frac{\varphi(c)}{c} - \frac{\varphi(a)}{a}\right) - \mathrm{rad}(abc)\frac{d}{a} = \mathrm{rad}(abc)\frac{\varphi(c)}{c} - \mathrm{rad}(abc)\frac{\varphi(a)}{a} - \mathrm{rad}(abc)\frac{d}{a},$$

and

$$c \,\Big|\, \mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(a)}{a}\right) - \mathrm{rad}(abc)\frac{d}{a} = \mathrm{rad}(abc)\frac{\varphi(b)}{b} - \mathrm{rad}(abc)\frac{\varphi(a)}{a} - \mathrm{rad}(abc)\frac{d}{a},$$

Thus, using *Lemma* 8,

$$\begin{aligned} L(a) &\leqq L\left(\mathrm{rad}(abc)\frac{\varphi(b)}{b} - \mathrm{rad}(abc)\frac{\varphi(c)}{c}\right) \\ &\leqq 2 \cdot L\left(\mathrm{rad}(abc)\frac{\varphi(b)}{b}\right) \\ &\leqq 2 \cdot L(\mathrm{rad}(abc)) \end{aligned}$$

and using *Lemma* 9,

$$\begin{aligned} L(b) &\leqq L\left(\mathrm{rad}(abc)\frac{\varphi(c)}{c} - \mathrm{rad}(abc)\frac{\varphi(a)}{a} - \mathrm{rad}(abc)\frac{d}{a}\right) \\ &\leqq 2 \cdot L\left(\mathrm{rad}(abc)\frac{\varphi(c)}{c}\right) \\ &\leqq 2 \cdot L(\mathrm{rad}(abc)) \end{aligned}$$

$$\begin{aligned} L(c) &\leqq L\left(\mathrm{rad}(abc)\frac{\varphi(b)}{b} - \mathrm{rad}(abc)\frac{\varphi(c)}{c} - \mathrm{rad}(abc)\frac{d}{a}\right) \\ &\leqq 2 \cdot L\left(\mathrm{rad}(abc)\frac{\varphi(b)}{b}\right) \\ &\leqq 2 \cdot L(\mathrm{rad}(abc)). \end{aligned}$$

□

Proposition 13: Let $a, b, c \in \mathcal{N}$ be relatively prime numbers satisfying $a + b = c$ and $\varphi(a) + \varphi(b) > \varphi(c)$. If $a = \mathrm{rad}(a)$ and $L(\varphi(a) + \varphi(b) - \varphi(c) \leqq L(a)$ then

$$\max(L(a), L(b), L(c)) \leqq 2 \cdot L(\mathrm{rad}(abc))$$

Proof.

We may assume that $L(a), L(b), L(c) \geqq 3$ by Proposition 9 and that $\varphi(a) + \varphi(b) - d = \varphi(c)$, where $d > 0$.

By the proof of *Proposition* 11, we see that

$$\frac{b}{a} = \frac{\frac{\varphi(c)}{c} - \frac{\varphi(a)}{a} + \frac{d}{a}}{\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}} = \frac{\mathrm{rad}(abc)\left(\frac{\varphi(c)}{c} - \frac{\varphi(a)}{a}\right) + \mathrm{rad}(abc)\frac{d}{a}}{\mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}\right)},$$

$$\frac{c}{a} = \frac{\frac{\varphi(b)}{b} - \frac{\varphi(a)}{a} + \frac{d}{a}}{\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}} = \frac{\mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(a)}{a}\right) + \mathrm{rad}(abc)\frac{d}{a}}{\mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}\right)}.$$

Hence, since $\mathrm{rad}(a) = a$

$$\mathrm{rad}(abc)\frac{d}{a} \in \mathcal{N}.$$

Moreover, since $\frac{b}{a}$ and $\frac{c}{a}$ are irreducible,

$$a \,\Big|\, \mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(c)}{c}\right) = \mathrm{rad}(abc)\frac{\varphi(b)}{b} - \mathrm{rad}(abc)\frac{\varphi(c)}{c},$$

$$b \,\Big|\, \mathrm{rad}(abc)\left(\frac{\varphi(c)}{c} - \frac{\varphi(a)}{a}\right) + \mathrm{rad}(abc)\frac{d}{a} = \mathrm{rad}(abc)\frac{\varphi(c)}{c} - \mathrm{rad}(abc)\frac{\varphi(a)}{a} + \mathrm{rad}(abc)\frac{d}{a},$$

and

$$c \,\Big|\, \mathrm{rad}(abc)\left(\frac{\varphi(b)}{b} - \frac{\varphi(a)}{a}\right) + \mathrm{rad}(abc)\frac{d}{a} = \mathrm{rad}(abc)\frac{\varphi(b)}{b} - \mathrm{rad}(abc)\frac{\varphi(a)}{a} + \mathrm{rad}(abc)\frac{d}{a},$$

Thus, using *Lemma* 8,

$$\begin{aligned} L(a) &\leqq L\left(\mathrm{rad}(abc)\frac{\varphi(b)}{b} - \mathrm{rad}(abc)\frac{\varphi(c)}{c}\right) \\ &\leqq 2 \cdot L\left(\mathrm{rad}(abc)\frac{\varphi(b)}{b}\right) \\ &\leqq 2 \cdot L(\mathrm{rad}(abc)) \end{aligned}$$

and using *Corollary* 10, since $L(d) \leqq L(a)$

$$\begin{aligned} L(b) &\leqq L\left(\mathrm{rad}(abc)\frac{\varphi(c)}{c} - \mathrm{rad}(abc)\frac{\varphi(a)}{a} + \mathrm{rad}(abc)\frac{d}{a}\right) \\ &\leqq 2 \cdot \max\left(L\left(\mathrm{rad}(abc)\frac{\varphi(c)}{c}\right), L\left(\mathrm{rad}(abc)\frac{d}{a}\right)\right) \\ &\leqq 2 \cdot L(\mathrm{rad}(abc)) \end{aligned}$$

$$\begin{aligned} L(c) &\leqq L\left(\mathrm{rad}(abc)\frac{\varphi(b)}{b} - \mathrm{rad}(abc)\frac{\varphi(c)}{c} + \mathrm{rad}(abc)\frac{d}{a}\right) \\ &\leqq 2 \cdot \max\left(L\left(\mathrm{rad}(abc)\frac{\varphi(b)}{b}\right), L\left(\mathrm{rad}(abc)\frac{d}{a}\right)\right) \\ &\leqq 2 \cdot L(\mathrm{rad}(abc)). \end{aligned}$$

□

Proposition 14: There are many counterexmples for

$$\max(L(a), L(b), L(c)) \leqq L(\mathrm{rad}(abc)).$$

Example 1:

a	b	c	rad(abc)	$L(a)$	$L(b)$	$L(c)$	L(rad)
1	8	9	6	0	3	2	2
5	27	32	30	2	3	5	4
32	49	81	42	5	4	4	4
3	125	128	30	1	6	7	4
7	121	128	154	2	6	7	6
3125	6859	9984	∗ ∗ ∗	9	11	12	11
6591	83521	90112	∗ ∗ ∗	10	16	16	12

Now, from the results of *Propsition*s 5, 11, 12 and 13, we would propose to state the following conjecture.

Conjecture : Let $a, b, c \in \mathcal{N}$ be relatively prime numbers satisfying $a + b = c$. Then

$$\max(L(a), L(b), L(c)) \leqq 2 \cdot L(\mathrm{rad}(abc)).$$

This *abc Conjecture* for a derived logarithmic function L of the Euler function φ is correct, the proof of Fermat's Last theorem becomes much shorter and easier as follows:

We assume that the co-prime x^p, y^p, $z^p \in \mathcal{N}$ satisfy $x^p + y^p = z^p$.

If our conjecture is correct,

$$\begin{aligned} pL(x) = L(x^p) &\leqq 2 \cdot L(\mathrm{rad}(x^p y^p z^p)) \\ &= 2 \cdot L(\mathrm{rad}(xyz)) \\ &\leqq 2 \cdot L(xyz) = 2(L(x) + L(y) + L(z)) \end{aligned}$$

$$\begin{aligned} pL(y) = L(y^p) &\leqq 2 \cdot L(\mathrm{rad}(x^p y^p z^p)) \\ &= 2 \cdot L(\mathrm{rad}(xyz)) \\ &\leqq 2 \cdot L(xyz) = 2(L(x) + L(y) + L(z)) \end{aligned}$$

$$\begin{aligned} pL(z) = L(z^p) &\leqq 2 \cdot L(\mathrm{rad}(x^p y^p z^p)) \\ &= 2 \cdot L(\mathrm{rad}(xyz)) \\ &\leqq 2 \cdot L(xyz) = 2(L(x) + L(y) + L(z)) \end{aligned}$$

Therefore

$$p(L(x) + L(y) + L(z)) \leqq 6(L(x) + L(y) + L(z)),$$

hence $p \leqq 6$.

But for exponents $n = 3, 4, 5, 6$ we already have proofs, which were proved by Fermat, Euler, Dirichlet or Legendre, so no three positive integers x, y, z such that $x^p + y^p = z^p$ for $p > 2$.

IV. Conclusion

In this paper, we review the derived logarithmic function of Euler function and its extension which are defined by one of authors. And also describe some nice linear property of the function.

We perform some kind of calculation related to so-called the "*abc*" problem on the derived logarithmic function by using computer applications, and verified it up to a certain bound.

We also prove the same type of problem under a certain condition. Finally state an "*abc*" problem as a open conjecture.

References

[1] Kurokawa, N., Koyama, S., *Introduction to the abc conjecture (in Japanese)*, PHP Institute, Inc., (2013)

[2] Parab, A., The *ABC*-conjecture for polynomials, http://www.math.purdue.edu/˜egoins/seminar/12-09-14.pdf, ref.2015.2.28

[3] Miyata, D., Yamasita, M. :Note on derived logarithmic functions of Euler's functions, *Proceedings of Autum meeting(App. Math.)*, Math. Soc. of Japan, http://yamashita-lab.net/open/mathconf-0.pdf, 2004.9, (in Japanese)

[4] Shapiro, Harold N., *Introduction to the Theory of Numbers*, John Wiley & Sons, New York et al.,(1983) [3. Arithmetic Functions §3.7 The Euler Function. Exrcise] 17 (p.77-78)]

[5] Murányi, Aladár, Az Euler-félé ϕ -függvény iterálásával nyert számelméleti függvényröl, Mat. Lapok 11 (1960), 47-67

[6] Shapiro, H., An arithmetic function arising from the ϕ function, *Amer. Math. Monthly*, 50 (1943), 18-30

[7] Pillai Sivasankaranarayana S., On some functions connected with $\phi(n)$, *Bull. Amer. Soc.*, 35 (1929), 832-836

[8] Pillai Sivasankaranarayana S., On a function connected with $\phi(n)$, *Bull. Amer. Soc.*, 35 (1929), 837-841